MARKED LENGTH SPECTRUM RIGIDITY IN NONPOSITIVE CURVATURE WITH SINGULARITIES

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ABSTRACT. Combining several previously known arguments, we prove marked length spectrum rigidity for surfaces with nonpositively curved Riemannian metrics away from a finite set of cone-type singularities with cone angles $> 2\pi$. With an additional condition, we can weaken the requirement on one metric to 'no conjugate points.'

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold (perhaps with a small set of singularities). For any element $[\gamma]$ of $\pi_1(M)$ there is a (not necessarily unique) shortest length (piecewise) geodesic γ in M representing $[\gamma]$. The function $l_g : \pi_1(M) \to \mathbb{R}$ which assigns the length of γ to $[\gamma]$ is called the *marked length spectrum* of (M, g). One may then ask the following general question:

Question 1.1 (The marked length spectrum rigidity question)**.** *To what extent does the function l_g determine the geometry of (M, g)?*

This is a well-studied problem; we will note some of the the results on it below. This paper addresses it for a class of singular metrics on surfaces.

Let S be a compact, connected, orientable surface without boundary. Suppose that there is a metric on S and a finite set of points $P \subset S$ such that on $S \smallsetminus P$ the metric is induced by a Riemannian metric. We call a point $p \in P$ a *cone-type singularity* with *cone angle* α if

$$\alpha = \lim_{r \to 0} \frac{L_r}{r}$$

where L_r is the length of the circle of radius r around p. Note that a cone point with angle $> 2\pi$ behaves like a concentrated point of negative curvature – around such a point the length of the circles of radius r grow faster than they do in the Euclidean case.

We say a metric on S is a *nonpositively curved cone metric* if the metric is Riemannian with nonpositive curvature on $S \smallsetminus P$, and at each point in P there is a cone-type singularity with cone angle $> 2\pi$. Let $\mathscr{M}_{npc}(S)$ denote the set of nonpositively curved cone metrics on S. We say that a metric on S is a *no conjugate points cone metric* if it has a finite set of cone points P with cone angles $> 2\pi$ and on $S \smallsetminus P$ the metric is Riemannian without conjugate points. We denote the set of no conjugate points cone metrics on S by $\mathscr{M}_{ncpc}(S)$ and note that $\mathscr{M}_{npc}(S) \subset \mathscr{M}_{ncpc}(S)$; examples first constructed by Gulliver ([Gul75]) show that the containment is strict (adapting his constructions to allow for cone points is straightforward).

A *strip* is the homeomorphic image of a map from $\mathbb{R} \times [0, \epsilon]$ into S where the first coordinate parametrizes unit-speed geodesics, and the lifts of these geodesics to $\tilde{S}$ remain at a bounded distance from one another. In non-positive curvature, after re-parametrization such a strip is necessarily an isometric immersion – a *flat strip*. The goal of this paper is to prove the following theorem:

Theorem 1. *Let S be a surface of genus ≥ 2. Let $g_1 \in \mathscr{M}_{ncpc}(S)$ and $g_2 \in \mathscr{M}_{npc}(S)$, and assume that $l_{g_1} = l_{g_2}$. Assume in addition that g_1 satisfies the following condition:*

$$vol(\{v : v \text{ is tangent to a geodesic lying in a strip}\}) = 0$$

where vol denotes the volume form $\lambda \wedge d\lambda$, where $\lambda = pdx$ is the Liouville form on the unit tangent bundle of $S \setminus P_{g_1}$ where P_{g_1} is the set of singularities for g_1. Then (S, g_1) is isometric to (S, g_2) by an isometry isotopic to the identity.

I do not know whether the assumption on the volume of strips is redundant; i.e. whether this is true of all metrics without conjugate points. We will show below that it is satisfied if g_1 is in fact in $\mathscr{M}_{npc}(S)$, and in proving this we will see that the existence of cone points is not the issue, so this is a question only about non-singular metrics without conjugate points on a surface with genus > 2.

If we restrict attention to metrics in $\mathscr{M}_{npc}(S)$, we can drop the strip volume assumption as well as the assumption on genus since the only nonpositively curved cone metrics on the torus are the flat metrics, where rigidity is clear.

Corollary 2. *Let S be any surface and let $g_1, g_2 \in \mathscr{M}_{npc}(S)$. Assume that $l_{g_1} = l_{g_2}$. Then (S, g_1) is isometric to (S, g_2) by an isometry isotopic to the identity.*

The proof of Theorem 1 combines ideas from the previous work of a number of authors on the marked length spectrum rigidity problem for surfaces. The general problem was posed in [BK85, §3]. Shortly thereafter it was proved for surfaces of negative curvature by Otal [Ota90] and, independently and at roughly the same time, by Croke [Cro90]. The methods in this paper mainly follow Otal's work. The result was extended to g_1 without conjugate points and g_2 nonpositively curved with a small (empty interior) region of zero curvature by Fathi [Fat89]. Croke, Fathi, and Feldman extended this result to g_2 of general nonpositive curvature in [CCF92].

Hersonsky and Paulin [HP97] proved MLS rigidity for negatively curved metrics with finitely many cone point singularities of angle $> 2\pi$. Duchin, Leininger and Rafi proved it for metrics coming from quadratic differentials [DLR10] – these are a special subset of the locally Euclidean metrics with cone points of angle $> 2\pi$. Recently, Bankovic and Leininger [BL15] have extended this result to all piecewise Euclidean metrics with cone singularities of angle $> 2\pi$. At the end of their paper they asked whether their ideas could be combined with those of Croke, Fathi, and Feldman to prove rigidity for nonpositively curved metrics with cone points. Frazier proved that the marked length spectrum distinguishes between the various curvature settings of these results [Fra12].

There are a few marked-length spectrum rigidity results for higher-dimensional manifolds ([Ham90], [DK02]) and non-manifold spaces ([CL12]). In general, the problem is very much open for dimension greater than two.

The proof of Theorem 1 consists of combining the ideas from this sequence of papers. We follow the approach via geodesic currents initiated by Otal. As in Croke,

Fathi, and Feldman's work, much of this approach works under the no conjugate points and nonpositive curvature assumptions on g_1 and g_2, respectively, and we follow Hersonsky and Paulin in extending Otal's ideas 'measurably' to a setting with singularities. A key step in the argument is the definition of the function $\theta'(v, \theta)$ (see §6 below) where one essentially needs to be able to detect which geodesics in the metrics contain cone points. Hersonsky and Paulin handle this using the Möbius current, which they develop and use to prove other results; it is not clear that this can be made to work in the current setting. Instead, we use a result of Bankovic and Leininger which shows that cone points can be detected in a more 'low tech' way – by looking at the support of a certain geodesic current. These extensions of Otal's methods provide an isometry between the sets of points in the two manifolds at which the curvature is negative, or which are cone points. We then adapt some more ideas from Croke, Fathi, and Feldman, and from Bankovic and Leininger to extend the isometry to the full surface.

2. Geodesics for metrics in $\mathscr{M}_{ncpc}(S)$

In this section we prove a few results on the structure of geodesics in $\tilde{S}$ for a metric with no conjugate points and with cone points of angle $> 2\pi$. The results are straightforward under an assumption of nonpositive curvature, so the reader only interested in that case can skip this section apart from the definitions. They are also known in the absence of cone points, see [Gre54].

Let $\tilde{g}$ denote the lift of the metric g to the universal cover $\tilde{S}$ of S. We will call a $\tilde{g}$-geodesic *non-singular* if it does not hit any cone points for $\tilde{g}$.

We want to prove that the exponential map, where defined, is injective. This implies that any $\tilde{g}$-geodesic is length-minimizing, and will be used to show that $\tilde{g}$-geodesics do not intersect, separate, and then intersect again. For complete manifolds without conjugate points and without cone points, this is Hadamard's theorem (see, e.g. [dC92, Thm 3.1]). Here we adapt aspects of that proof to deal with large-angle cone points.

Let $\exp_p^{\tilde{g}}$ (or $\exp_p$ when the metric is understood) denote the exponential map from $T_p\tilde{S}$ to $\tilde{S}$. This is defined for nonsingular points p, but it is clear that we can allow cone points ζ if we interpret $T_\zeta\tilde{S}$ as the set of nonnegative multiples of elements in the space of directions at ζ (see [BH99, Defn II.3.18] for a definition of the space of directions). Because of the singular points, $\exp_p$ may not be defined for all vectors in $T_p\tilde{S}$. Let $\mathscr{V}_p$ be the set of all points in $\tilde{S}$ which can be reached from p by a geodesic which hits no cone points (except perhaps p itself). Let V_p be such that $\exp_p(V_p) = \mathscr{V}_p$ and which is *star-shaped*: $v \in V_p$ implies $\lambda v \in V_p$ for all $\lambda \in [0, 1]$. Note that $\mathscr{V}_p$ and V_p are open.

Lemma 2.1. $\exp_p : V_p \to \mathscr{V}_p$ *is a covering map.*

Proof. The proof is the same as the classical (no singularities) case. See, e.g. [dC92, Lemma 3.3]. □

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Lemma 2.2. *Let $\mathscr{U}$ be a simply connected subset of $\mathscr{V}_p$ containing p. Suppose that every point of $\mathscr{U}$ can be reached from p by a geodesic in $\mathscr{U}$. Then there is a star-shaped $U \subset V_p$ containing 0 such that $\exp_p(U) = \mathscr{U}$ and $\exp_p$ is injective on U.*

Proof. Again, the proof is the same as in the classical case. Use Lemma 2.1 and pick for U the connected component of $\exp_p^{-1}(\mathscr{U})$ containing 0. It is clearly star-shaped. □

Lemma 2.3. $\exp_p$ *is injective on* V_p.

Proof. Consider $\mathscr{U}_r = \exp_p(U_r)$ where $U_r = V_p \cap B_r(0)$. Note that every point in $\mathscr{U}_r$ can be reached from p by a geodesic entirely in $\mathscr{U}$, since U_r is star-shaped.

By Lemma 2.2, any failure of injectivity for $\exp_p$ must arise from a situation where $\exp_p(U_r)$ is not simply connected. This can only happen if two radial geodesics from p meet after going around opposite sides of some cone point ζ. This situation is depicted in Figure 1.
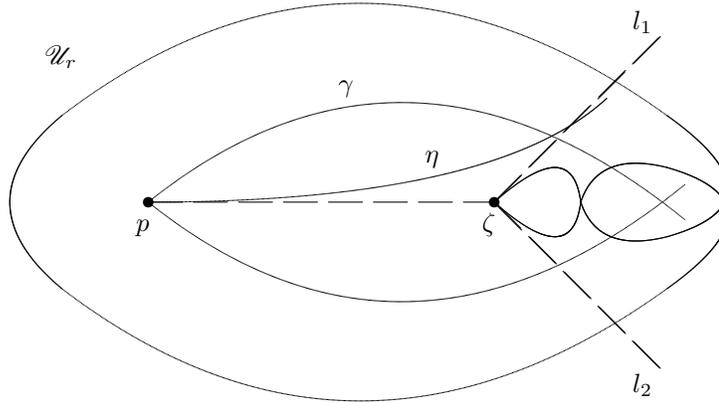

FIGURE 1. Potential non-injectivity of $\exp_p$ 'behind' a cone point ζ.

In Fig 1, l_1 and l_2 are geodesics from p through the cone point ζ making angle π on either side of ζ. A pair of geodesics demonstrating potential non-injectivity of the exponential map 'behind' ζ are shown; we see that $\exp_p(U_r)$ is not simply connected, due to the cone point. It is clear that one of the two intersecting geodesics, say γ, intersects l_1 or l_2, say l_1. Take a tangent vector in V_p at p very close to the vector between p and ζ and on γ's side of ζ. The geodesic η it generates must stay close to l_1 for a long time, and therefore must intersect γ before it leaves $\mathscr{U}_r$.

Restrict U_r to those vectors generating the top half of Figure 1. If the image of these vectors under the exponential map is simply connected, the intersection of γ and η is a contradiction to Lemma 2.2. If not, there must be some other cone point $\zeta' \neq \zeta$ between γ and η. Apply the argument above again to find a pair of distinct intersecting geodesics in $\mathscr{U}_r$ which enclose a region containing neither ζ nor ζ'. Continuing this way, and noting that for any r, there are at most finitely many cone points within distance r of p, we eventually reach a contradiction to Lemma 2.2, completing the proof.

□

Proposition 2.4. *The intersection of two distinct $\tilde{g}$-geodesics in $\tilde{S}$ for $g \in \mathscr{M}_{ncpc}(S)$ has at most one connected component.*

This has an immediate corollary:

Corollary 2.5. *No $\tilde{g}$-geodesic self-intersects, and every $\tilde{g}$-geodesic is minimizing.*

Proof of Proposition 2.4. Suppose we have two geodesics l_1 and l_2 which intersect at p and q, but not between. By Lemma 2.3, at least one of these geodesics is singular and hits a cone point between p and q. We will prove the result by induction on the number of cone points which lie between l_1 and l_2 between p and q or which lie on them but make angle $> \pi$ on the 'between' side.

Suppose there is only one such cone point, ζ on l_1 or l_2 between their intersections making angle $> \pi$ on the 'between' side. Assume ζ belongs to l_1. Let l_1' be the geodesic through p and ζ making angle π on the side of ζ to which the segment of l_2 between p and q lies (see Figure 2).
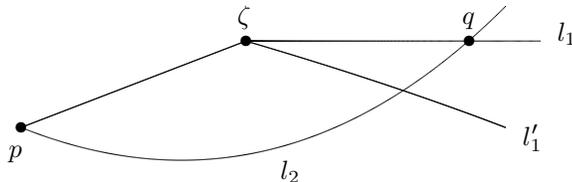

FIGURE 2. Base case for the induction in Propostion 2.4.

We see then that l_1' and l_2 intersect before l_2 reaches q. We know that l_2 is non-singular over this segment. Although l_1' is not, we may approximate it by a non-singular segment on the l_2-side of ζ since the angles along l_1 on that side are always π. The intersection of this approximating segment and l_2 yields a contradiction to Lemma 2.3.

Now suppose we have proven the result when n or fewer cone points lie between l_1 and l_2. We can reduce to the case where all the cone points lie on the geodesics by re-drawing l_1 from p through a cone point between l_1 and l_2. At the last cone point before q on l_1 or l_2 where an angle $> \pi$ is made, replace the geodesic by the angle π geodesic on the correct side of that cone point as above. We see that the number of cone points which are between l_1 and l_2 and at which one of the geodesics makes angle $> \pi$ strictly decreases. By the inductive hypothesis, we are done. □

We will need the following below. An analogous result is the key to Fathi and Croke-Fathi-Feldman's extension to the 'no conjugate points' assumption for g_1. They reference Morse ([Mor24]) for the version without cone points; we give an alternate proof here allowing cone points.

Proposition 2.6. *Let $g_1, g_2 \in \mathscr{M}_{ncpc}(S)$ for a surface of genus ≥ 2. Then there exists a constant $K > 0$, depending only on g_1 and g_2, such that any $\tilde{g}_1$-geodesic contains a $\tilde{g}_2$-geodesic in its K-neighborhood, and vice versa.*

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Proof. Let $\Gamma = \pi_1(S)$. Each of $(\tilde{S}, \tilde{g}_1)$ and $(\tilde{S}, \tilde{g}_2)$ are quasi-isometric to Γ with a word metric, and hence are δ-hyperbolic metric spaces for some δ. There is a quasi-isometry $(\tilde{S}, \tilde{g}_1) \to (\tilde{S}, \tilde{g}_2)$. Since any $\tilde{g}_1$-geodesic is minimizing (by Corollary 2.5), it is a quasi-geodesic for $\tilde{g}_2$. The result follows from the stability of quasi-geodesics in δ-hyperbolic spaces (see, e.g., [BH99, III.H Theorem 1.7]). $\qquad\square$

If S is endowed with a metric g in $\mathscr{M}_{ncpc}(S)$, then $(\tilde{S}, \tilde{g})$ can be compactified by adding the boundary at infinity $\partial^\infty(\tilde{S})$, which consists of equivalence classes of asymptotic geodesic rays. Note that since $\Gamma := \pi_1(S)$ acts cocompactly on $\tilde{S}$, we can identify $\partial^\infty(\tilde{S})$ with $\partial^\infty(\Gamma)$ where Γ is endowed with the word metic for any finite generating set.

By Corollary 2.5 all geodesics for $\tilde{g} \in \mathscr{M}_{ncpc}(S)$ are minimizing, so to any geodesic l in $(\tilde{S}, \tilde{g})$ we can associate $l(+\infty)$ and $l(-\infty)$, its forward and backward endpoints in $\partial^\infty(\tilde{S})$. If g has strictly negative curvature, any pair of distinct points $\{\xi_1, \xi_2\}$ in $\partial^\infty(\tilde{S})$ specifies a unique geodesic in $(\tilde{S}, \tilde{g})$. In nonpositive curvature, any two geodesics in the universal cover with the same endpoints in $\partial^\infty(\tilde{S})$ bound a *flat strip*, that is, an isometrically embedded copy of $\mathbb{R} \times [a, b]$ ([Gre54, Thm 4.1]). With only the no conjugate points assumption, they bound a (not necessarily flat) strip and through each (non-cone) point in this strip runs a unique geodesic with the same endpoints at infinity ([Gre54, Cor 3.1]).

Definition 2.7. *Let $\mathcal{G}_{\tilde{g}}$ be the set of images of $\tilde{g}$-geodesics. We topologize $\mathcal{G}_{\tilde{g}}$ by considering it as the quotient space of the space of* parametrized $\tilde{g}$-geodesics with the *topology of convergence on compact sets, where we form the quotient by forgetting the parametrization.*

Let

$$\mathscr{G}(\tilde{S}) = [(\partial^\infty(\tilde{S}) \times \partial^\infty(\tilde{S})) \smallsetminus \Delta]/((x, y) \sim (y, x))$$

(where Δ is the diagonal) be the set of unordered pairs of distinct points in the boundary of $\tilde{S}$.

Definition 2.8. *Let $\partial_{\tilde{g}} : \mathcal{G}_{\tilde{g}} \to \mathscr{G}(\tilde{S})$ send a geodesic l to $\{l(+\infty), l(-\infty)\}$.*

Note that $\mathcal{G}_{\tilde{g}}$ depends on the metric, while $\mathscr{G}(\tilde{S})$ depends only on Γ. Due to Proposition 2.6 (applied with g_2 of strictly negative curvature, for instance) $\partial_{\tilde{g}}$ is surjective. As noted above, it is injective away from geodesics contained in strips.

Definition 2.9. *We say $\tilde{g}_1$-geodesic l corresponds to $\tilde{g}_2$-geodesic l' if $\partial_{\tilde{g}_1}(l) = \partial_{\tilde{g}_2}(l')$.*

Definition 2.10. *Let $\mathcal{G}_{\tilde{g}}^\circ$ be the set of non-singular $\tilde{g}$-geodesics, i.e. those not hitting any cone point. Let $\mathcal{G}_{\tilde{g}}^* = \overline{\mathcal{G}_{\tilde{g}}^\circ}$.*

We now want to prove that $\mathcal{G}_{\tilde{g}}^\circ$ is non-empty, and to characterize what will turn out to be almost every geodesic in $\mathcal{G}_{\tilde{g}}^*$ (see Lemma 5.2). The first will be essential for our construction of the Liouville current (Section 4); the second is essential to use Bankovic-Leininger's characterization of the support of that current (Proposition 5.3).

Proposition 2.11. *Let $g \in \mathscr{M}_{ncpc}(S)$. At any non-cone point $p \in \tilde{S}$ the set of geodesics through p in $\mathcal{G}_{\tilde{g}}^\circ$ is full measure with respect to the angular measure.*

Proof. Two distinct vectors based at p generate geodesics that never intersect elsewhere, by Proposition 2.4, so each cone point lies on the geodesic generated by at most one such vector. There are only countably many cone points in $\tilde{S}$, so the proposition follows. □

Proposition 2.12 (cf. Prop 2.3, [BL15]). *Let $g \in \mathcal{M}_{ncpc}(S)$. If l is a $\tilde{g}$-geodesic containing at most one cone point and making angle π on one side at that point, then $l \in \mathcal{G}_{\tilde{g}}^*$.*

Proof. Let ζ be the cone point contained in l. Let v^* be the tangent vector to l at ζ; again, at singular points we interpret $T_\zeta S$ as the set of nonnegative multiples of directions from p. For any sequence (v_n) of vectors tangent to geodesics (l_{v_n}) in $\mathcal{G}_{\tilde{g}}^\circ$ with v_n approaching v^* via basepoints on the π-side of l, the forward ray of l_{v_n} converges to the forward ray of l, since there are no cone points along that ray. The same is true for backward rays since there are again no cone points along those rays and the angle l makes at that side of ζ is π.

By Proposition 2.11 almost every vector for the Lebesgue measure on $T^1(\tilde{S} - \tilde{P})$ is tangent to a geodesic in $\mathcal{G}_{\tilde{g}}^\circ$. So a sequence answering the requirements of (v_n) above exists.

□

We close this section by proving a claim mentioned in the introduction – that the volume in $T^1(S \smallsetminus P)$ of flat strips for a metric in $\mathcal{M}_{npc}(S)$ is zero.

Proposition 2.13. *Let g be in $\mathcal{M}_{npc}(S)$ where S has genus at least 2, and let B be the set of all $v \in T_g^1 S$ which are based at a point in some flat strip, and point in the direction of the strip. Then $vol_g(B) = 0$ where vol denotes the usual measure on T_g^1.*

Proof. Note that no flat strip may contain a cone point, so the argument necessarily takes place in $S \smallsetminus P$.

Assume the contrary. Let $\mathcal{F}_g$ be the set of flat strips for g. We may assume that each strip in F is maximal, in the sense that it cannot be extended to a flat strip with greater width. As S has genus > 2, such a maximum width must exist. The volume in $T^1(S \smallsetminus P)$ of any single flat strip is zero, so if B is to have positive volume, there must exist some $\delta > 0$ such that there are infinitely many flat strips with width $> \delta$.

Let (F_j) be a sequence of distinct (maximal) flat strips with width $> \delta$. There is a subsequence (F_{i_j}) converging to a maximal flat strip G with width $> \delta$ in the sense that there is a tangent vector v_G to the geodesic direction to the strip G and a sequence of vectors $(v_{i_j}) \to v_G$ such that v_{i_j} is tangent to the geodesic direction at the center of the strip F_{i_j}. We may assume G does not belong to the sequence F_{i_j} by removing it if necessary. For sufficiently large j, F_{i_j} and G overlap, and so we can define the angle α_{i_j} between v_{i_j} and v_G using the fact that the strips are flat. We note that this angle cannot be zero, else $F_{i_j} \cup G$ would provide an extension of either F_{i_j} or G, contradicting maximality.

Now consider the flat half-strips, i.e. isometric immersions of $\mathbb{R}_+ \times [a,b]$ which are obtained from restricting G to $[L, \infty)$ in the geodesic coordinate, for any L. Again, each such flat half-strip is contained in a maximal width flat half-strip. Let W_{max} be the supremum of all the widths of these. Since the genus of S is at least 2, it is again clear that $W_{max} < \infty$. Pick some flat half-strip arising from G with

width at least $W_{max} - \frac{\delta}{16}$; call it G^+. Note that flat half-strips $F_{i_j}^+$ coming from the F_{i_j} approach G^+, and that the angles between these flat half strips (for sufficiently large j) are α_{i_j}.

Now it is possible to use an argument of Cao and Xavier in [CX08] (also discussed in [CS14, Prop 3.1]) to produce from G^+ and the $F_{i_j}^+$ a flat half-strip along G with width $\frac{\delta}{8}$ greater than the width of G^+, producing a contradiction.

$\square$

Remark 2.14. The argument above also works under the weaker assumption that there are uncountably many maximal flat strips. The cone points play no role in the question – in fact once you have infinitely many strips with width at least δ, one could perform a surgery on $\frac{\delta}{3}$-neighborhoods of the cone points and produce a complete, nonpositively curved metric with infinitely many strips of width at least $\frac{\delta}{3}$ and proceed from there. Though Cao and Xavier's argument will not work under the weaker assumption of no conjugate points, the fact that we have not exploited the full strength of the positive volume assumption above gives some hope that the following question may have a positive answer.

Question 2.15. *Let g be a Riemannian metric on a closed surface of genus at least two without conjugate points. Does the set of all tangent vectors for strips have zero volume?*

3. Geodesic currents

Apart from the original work of Croke [Cro90], proofs of marked length spectrum rigidity for surfaces all rely on the fundamental work of Otal relating the marked length spectrum to geodesic currents. (See [Bon86, Bon88] for good references on geodesic currents.)

Definition 3.1. *A geodesic current on (S, g) is a Γ-invariant Radon measure on $\mathscr{G}(\tilde{S})$.*

Write $\mathscr{C}(S)$ for the set of geodesic currents on S. Again, this depends only on Γ, not the particular metric. We endow $\mathscr{C}(S)$ with the weak*-topology.

A first, and crucial, example is the following. Let γ be a closed geodesic on (S, g). Abusing notation slightly, we can also view γ as an element of the fundamental group Γ. The set of all lifts of γ to $(\tilde{S}, \tilde{g})$ maps under $\partial_{\tilde{g}}$ to a discrete, Γ-invariant set of points in $\mathscr{G}(\tilde{S})$. Denote by $\langle \gamma \rangle$ the current given by the counting measure on this set.

We now record a few fundamental facts about geodesic currents.

Theorem 3.2 (See [Bon86], Proposition 4.2)**.** *The set of real multiples of geodesic currents of the form $\langle \gamma \rangle$ for $\gamma \in \Gamma$ is dense in $\mathscr{C}(S)$.*

Note that the intersection of two geodesics in $\mathcal{G}_{\tilde{g}}$ can be detected by the ordering of their endpoints around the boundary at infinity. In particular, intersection is independent of the choice of metric.

Definition 3.3 (See [Bon86], §4.2)**.** *For $\gamma \in \Gamma$, let $\mathcal{I}(\tilde{\gamma})$ be the subset of $\mathcal{G}_{\tilde{g}}$ consisting of all $\tilde{g}$-geodesics intersecting $\tilde{\gamma}$, the axis for γ. Let $I(\tilde{\gamma})$ be any fundamental domain for the action of γ on $\mathcal{I}(\tilde{\gamma})$. For any $\mu \in \mathscr{C}(S)$ we define the* intersection number of μ and γ *by*

$$i(\mu, \langle \gamma \rangle) = \mu(\partial_{\tilde{g}}(I(\tilde{\gamma}))).$$

The following theorem of Otal is fundamental to the entire approach to MLS rigidity through currents.

Theorem 3.4 ([Ota90], Théorème 2). *For $\mu_1, \mu_2 \in \mathscr{C}(S)$, $\mu_1 = \mu_2$ if and only if $i(\mu_1, \langle \gamma \rangle) = i(\mu_2, \langle \gamma \rangle)$ for all $\gamma \in \Gamma$.*

The intersection number of a current μ and a closed geodesic γ, in the special case where $\mu = \langle \gamma' \rangle$, counts the number of intersections of the two closed geodesics γ and γ'. The intersection number can be extended to a symmetric, bilinear pairing $i(\mu_1, \mu_2)$ on $\mathcal{C}(S)$, but this is not necessary for the arguments of this paper. (See [Bon86, §III and §IV] for details on this extension.)

4. THE LIOUVILLE CURRENT

We follow Hersonsky-Paulin and Bankovic-Leininger in extending the Liouville current in the presence of cone points. It is defined as follows.

Let k be an oriented geodesic segment for $\tilde{g}$ not hitting any cone points. Let $\mathcal{G}_{\tilde{g}}^{\circ}(k) \subset \mathcal{G}_{\tilde{g}}$ be the set of $\tilde{g}$ geodesics transversally intersecting the interior of k which do not hit any cone point. We introduce coordinates (t, θ) on $\mathcal{G}_{\tilde{g}}^{\circ}(k)$ by letting $t(l)$ be the distance from the starting point of k to its intersection with $l \in \mathcal{G}_{\tilde{g}}^{\circ}(k)$ and by letting $\theta(l) \in (0, \pi)$ be the angle between the forward direction of k and l. Let $D^{\circ}(k) \subset [a, b] \times (0, \pi)$ be all the coordinate pairs arising this way. As there are only countably many cone points in $(\tilde{S}, \tilde{g})$, $D^{\circ}(k)$ is a full Lebesgue measure subset of $[a, b] \times (0, \pi)$ (using Proposition 2.11). The bijection $D^{\circ}(k) \to \mathcal{G}_{\tilde{g}}^{\circ}(k)$ allows us to push forward the measure defined on $D^{\circ}(k)$ by

$$\hat{L}_g = \frac{1}{2} \sin \theta \, d\theta \, dt$$

to a measure $\hat{L}_g(k)$ on $\mathcal{G}_{\tilde{g}}^{\circ}(k)$.

Cover $\mathcal{G}_{\tilde{g}}^{\circ}$, the set of $\tilde{g}$-geodesics not hitting any cone point, by sets of the form $\mathcal{G}_{\tilde{g}}^{\circ}(k)$. Let $G_{\tilde{g}}^{\circ}$ be the set of parametrized non-singular geodesics in $(\tilde{S}, \tilde{g})$. We then have

Proposition 4.1. *The measures $\hat{L}_g(k)$ are transverse invariant measures on the foliation of $G_{\tilde{g}}^{\circ}$ by geodesics, and therefore define a global measure on $\mathcal{G}_{\tilde{g}}^{\circ}$, denoted by $\hat{L}_g$. This measure is Γ-invariant.*

Proof. Γ-invariance follows from the definition of $\hat{L}_g$, provided the other parts of the proposition can be proven.

That the $\hat{L}_g(k)$ are transverse invariant measures on the foliation by non-singular geodesics follows from an argument pointed out by Hersonsky and Paulin [HP97, Prop 4.11]: applying a countable cutting procedure to the well-known analogous result without singularities. □

Definition 4.2. *The Liouville current L_g is the element of $\mathscr{C}(S)$ formed by extending $\hat{L}_g$ to all of $\mathcal{G}_{\tilde{g}}$ by setting $\hat{L}_g(\mathcal{G}_{\tilde{g}} \setminus \mathcal{G}_{\tilde{g}}^{\circ}) = 0$ and pushing forward $\hat{L}_g$ by $\partial_{\tilde{g}}$ to a measure on $\mathscr{G}(\tilde{S})$.*

The following proposition gives the key relationship between this current and the geometry of (S, g):

Proposition 4.3. *For any geodesic segment k and $\mathcal{I}(k)$ the set of all $\tilde{g}$-geodesics intersecting k transversally in its interior, $L_g(\partial_{\tilde{g}}(\mathcal{I}(k))) = length_g(k)$.*

Proof. This follows easily from integrating the local coordinate expression for $\hat{L}_g$. $\square$

Proposition 4.4. *For every $\gamma \in \Gamma$,*

$$i(L_g, \langle\gamma\rangle) = length_g(\gamma).$$

Proof. Every γ is realized by a finite union of geodesic segments. Then use Theorem 3.3 and Proposition 4.3. $\square$

Combining these results, we get the following important result linking the marked length spectrum to the Liouville currents.

Proposition 4.5. *Suppose that $g_1, g_2 \in \mathscr{M}_{ncpc}(S)$ have the same marked length spectrum. Then $L_{g_1} = L_{g_2}$.*

Proof. By Theorem 3.4 we need only check that $i(L_{g_1}, \langle\gamma\rangle) = i(L_{g_2}, \langle\gamma\rangle)$ for all $\gamma \in \Gamma$. But this is provided by Proposition 4.4 and equality of the marked length spectrum. $\square$

5. Cone points

In [BL15], Bankovic and Leininger provide a novel way to detect cone points using the Liouville current. They work in the setting of nonpositively curved Euclidean cone metrics – metrics which are locally Euclidean away from a finite set of cone points which have cone angles $> 2\pi$. The reader can easily use the results of Section 2 verify that the results in the first four sections of their paper hold for metrics in $\mathscr{M}_{ncpc}(S)$. The key steps are to replace their use of the CAT(0) assumption in a few places by results we proved in Section 2, and to replace references to 'flat strips' by 'strips,' using the work in [Gre54] noted above.

Their identification of cone points relies on carefully studying the support of L_g. We restate their main results here.

Definition 5.1. *Let $\mathcal{G}_{\tilde{g}}^2 \subset \mathcal{G}_{\tilde{g}}^*$ be the set of geodesics in $\mathcal{G}_{\tilde{g}}^*$ containing at least two cone points. Let $\mathcal{G}_{\tilde{g}}^*(\zeta)$ be the set of geodesics in $\mathcal{G}_{\tilde{g}}^*$ which contain the cone point ζ.*

Lemma 5.2 (See [BL15], Corollary 2.5)**.** *$\mathcal{G}_{\tilde{g}}^2$ is countable.*

Proof. Bankovic and Leininger argue that containing more than one cone point and yet still lying in $\mathcal{G}_{\tilde{g}}^*$, that is, being the limit of non-singular geodesics, forces a geodesic to make angle π on one side at every cone point with the side on which this angle lies switching at most once. As there are countably many cone points, the result follows. $\square$

From its definition, the following is clear:

Proposition 5.3. *$Supp(L_g) = \partial_{\tilde{g}}(\mathcal{G}_{\tilde{g}}^*)$.*

Then by examining the support of L_g, specifically special sequences of points in $\partial_{\tilde{g}}(\mathcal{G}_{\tilde{g}}^*)$, they prove the following theorem.

Theorem 5.4 ([BL15], see Prop 4.5 and work in section 4). *Let $g_1, g_2 \in \mathscr{M}_{npc}(S)$ have the same marked length spectrum. Then there is a Γ-equivariant, bijective isometry $F_c : cone(\tilde{g}_1) \to cone(\tilde{g}_2)$ (with respect to the metrics induced by $\tilde{g}_1$ and $\tilde{g}_2$) between the cone points of $(\tilde{S}, \tilde{g}_1)$ and those of $(\tilde{S}, \tilde{g}_2)$. Furthermore, this isometry is induced by a map $\mathcal{F}_c : \bigcup_\zeta \mathcal{G}^*_{\tilde{g}_1}(\zeta) \smallsetminus \Omega \to \mathcal{G}_{\tilde{g}_2}$, for Ω countable, which takes the set of geodesics in $\mathcal{G}^*_{\tilde{g}_1} \smallsetminus \Omega$ passing through a cone point ζ of $\tilde{g}_1$ to geodesics passing through the cone point $F_c(\zeta)$ of $\tilde{g}_2$ in such a way that $\mathcal{F}_c(l)$ corresponds to l.*

Bankovic and Leininger introduce the following notion to prove this theorem:

Definition 5.5 (See §4 of [BL15]). *For a countable subset $\Omega \subset \partial_{\tilde{g}} \mathcal{G}^*_{\tilde{g}}$. An (L_g, Ω)-chain is a sequence (x_i) of points in $\partial^\infty(\tilde{S})$ such that for all i,*

- $\{x_i, x_{i+1}\} \in supp(L_g) \smallsetminus \Omega$, *where Ω is countable, and*
- x_i, x_{i+1}, x_{i+2} *is counterclockwise ordered and the set of all points in $supp(L_g)$ between $\{x_i, x_{i+1}\}$ and $\{x_{i+1}, x_{i+2}\}$ (in the sense that a geodesic realizing this point is between geodesics realizing $\{x_i, x_{i+1}\}$ and $\{x_{i+1}, x_{i+2}\}$) is only $\{\{x_i, x_{i+1}\}, \{x_{i+1}, x_{i+2}\}\}$. (See [BL15, §2.3] for the precise definition of 'between.')*
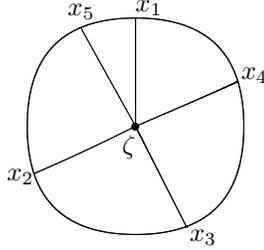
Figure 3 illustrates a few steps in an (L_g, Ω)-chain.

FIGURE 3. A few steps in an (L_g, Ω)-chain. The angle between $[\zeta, x_i]$ and $[\zeta, x_{i+1}]$ is π.

Outline of proof of Theorem 5.4. Let ζ be a cone point for $\tilde{g}$. Then there is a countable set Ω containing $\partial_\infty(\mathcal{G}^2_{\tilde{g}_i})$ for $i = 1, 2$ (thanks to Lemma 5.2), and a set of (L_g, Ω)-chains corresponding to geodesics passing through ζ. In addition, ζ is uniquely specified by these chains in that given any (L_g, Ω)-chain, it corresponds to a sequence of $\tilde{g}$-geodesics through a unique cone point ([BL15, Prop 4.1]). Furthermore, one can detect from conditions on $\partial^\infty(\tilde{S})$ alone whether two (L_g, Ω)-chains specify the same cone point ([BL15, Lemma 4.4]). As the marked length spectrum determines L_g, and hence $supp(L_g)$, equality of marked length spectra for g_1 and g_2 implies that we have a bijective map $cone(\tilde{g}_1) \to cone(\tilde{g}_2)$ (see [BL15, proof of Theorem 5.1]).

We now describe the map $\mathcal{F}_c$. Let ζ be a cone point for $\tilde{g}_1$. It is the unique cone point specified by an (L_{g_1}, Ω)-chain (see [BL15, Lemma 4.2]). The g_1-geodesics realizing this chain correspond to the $\tilde{g}_2$-geodesics doing the same. As noted above, these $\tilde{g}_2$-geodesics pass through $F_c(p)$. This proves the last statement of the Theorem.

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Finally, we want to show that F_c is an isometry. Let $\zeta_1, \zeta_2 \in cone(\tilde{g}_1)$, and let k be the $\tilde{g}_1$-geodesic segment between them. Consider the set $\mathcal{G}_{\tilde{g}_1}(k)$ of all $\tilde{g}_1$-geodesics l intersecting k transversally and such that $\partial_{\tilde{g}_1}(l) \notin \Omega$. Using the fact that F_c is induced by the map $\mathcal{F}_c$ described above and the fact that cone points cannot lie in the interior of a flat strip, it is not hard to see that the $\tilde{g}_2$-geodesics with the same endpoints as those in $\mathcal{G}_{\tilde{g}_1}(k)$ are precisely those intersecting the $\tilde{g}_2$ geodesic segment k' between $F_c(\zeta_1)$ and $F_c(\zeta_2)$ and not having endpoints in Ω. As Ω is countable, the set of geodesics with endpoints in Ω is measure zero for any Liouville current. Then using Propositions 4.3 and 4.5,

$$
\begin{aligned}
d_{\tilde{g}_1}(\zeta_1, \zeta_2) &= L_{g_1}(\partial_{\tilde{g}_1}(\mathcal{I}(k))) = L_{g_1}(\partial_{\tilde{g}_1}(\mathcal{G}_{\tilde{g}_1}(k))) \\
&= L_{g_2}(\partial_{\tilde{g}_2}(\mathcal{G}_{\tilde{g}_2}(k'))) = L_{g_2}(\partial_{\tilde{g}_2}(\mathcal{I}(k'))) \\
&= d_{\tilde{g}_2}(F_c(\zeta_1), F_c(\zeta_2)).
\end{aligned}
$$

This completes the proof.

$\square$

Remark 5.6. As we will see below, the work of Fathi and Croke-Fathi-Feldman (using Otal's ideas) produces a similar isometry between points where the curvature is negative. Since cone points with angle $> 2\pi$ are like point masses of negative curvature (as in the Gauss-Bonnet theorem with cone points), it is not surprising that this result would hold. What is particularly nice about Bankovic and Leininger's approach is its comparatively 'low-tech' approach – they only need to know the support of the Liouville current.

Let T_g^1 be the subset of $T^1(\tilde{S} - \tilde{P})$ consisting of unit tangent vectors for non-singular $\tilde{g}$-geodesics. Note that almost every vector in $T^1(\tilde{S} - \tilde{P})$ (with respect to the usual volume) is in T_g^1.

Proposition 5.7. *There is a volume zero, geodesic flow-invariant subset B of $T_{g_1}^1$ such that the following is true. Let l be a non-singular $\tilde{g}_1$-geodesic whose tangent vectors do not lie in B. Then no $\tilde{g}_2$-geodesic corresponding to l is singular. Specifically, B is the set of tangent vectors for non-singular geodesics which do not belong to a strip bordering a cone point.*

Proof. From Theorem 5.4 we know that any cone point for $\tilde{g}_2$ is $F_c(\zeta)$ for ζ some cone point of $\tilde{g}_1$. Under our assumption, ζ does not belong to l.

Let r_1 and r_2 be the geodesic rays from ζ to the endpoints $\partial_{\tilde{g}_1}(l) = \{x, y\}$. If r_1 and r_2 make angle $< \pi$ at ζ, then it is not hard to see that there is an (L_{g_1}, Ω)-chain through ζ containing the endpoints of a geodesic γ through ζ which makes angle π on the l side of ζ, and with endpoints distinct from x and y and on the ζ side of x and y. (See Figure 4.) The corresponding geodesic passing through $F_c(\zeta)$ for the (L_{g_2}, Ω)-chain has the same endpoints and also makes angle π on the side of $F_c(\zeta)$ to which x and y lie. We then see that any $\tilde{g}_2$-geodesic connecting x and y cannot pass through $F_c(\zeta)$, as desired.
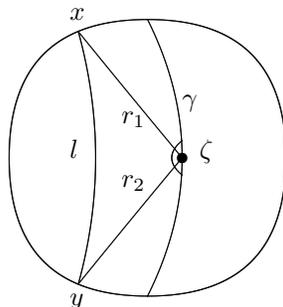
FIGURE 4. An (L_{g_1}, Ω)-chain geodesic γ through ζ entirely to one side of l. The marked angle is π. There are uncountably many such γ that can be drawn, so we can find one that avoids Ω.

Now suppose that r_1 and r_2 make angle π on the l-side of ζ. Then l and $r_1 \cup r_2$ bound a strip. Letting B be the set of tangent vectors to non-singular geodesics belonging to strips bordering a cone point gives the result, using either Proposition 2.13 if $g_1 \in \mathscr{M}_{npc}(S)$ or the assumptions of Theorem 1 if not. $\qquad \square$

Definition 5.8. *Let* $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1} \subset \mathcal{G}^{\circ}_{\tilde{g}_1}$ *be the set of all non-singular $\tilde{g}_1$-geodesics which do not belong to a strip bordering a cone point (as in Proposition 5.7). Let* $\hat{T}^1_{g_1}$ *be the set of all tangent vectors to geodesics in* $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$.

Proposition 5.7 shows that no geodesic corresponding to l is singular, so in fact no geodesic corresponding to l can lie in a strip bordering a cone point. We thus have

Corollary 5.9. *Geodesics in* $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$ *correspond only to geodesics in* $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_2}$ *and vice versa.*

6. Angle correspondence

The following result is the key to proving a version of Theorem 5.4 for points of negative curvature.

Proposition 6.1. *There is a full measure set of $\theta \in (0, \pi)$ such that the following holds. Suppose that l_1, l_2 are $\tilde{g}_1$-geodesics in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$ intersecting with angle θ. Then any two corresponding $\tilde{g}_2$-geodesics l'_1, l'_2 intersect with angle θ.*

Remark 6.2. Note that the angle between l'_1 and l'_2 is well-defined. Indeed, by definition of $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$ and Proposition 5.7, the l'_i are non-singular. Hence they intersect in a well-defined angle, and the problematic configurations in Figure 5 do not occur. Further, if l'_1 and l''_1 both correspond to l_i, then they bound a flat strip. The fact that this strip is flat ensures that the angle between l'_1 and l'_2 is the same as the angle between l''_1 and l'_2.
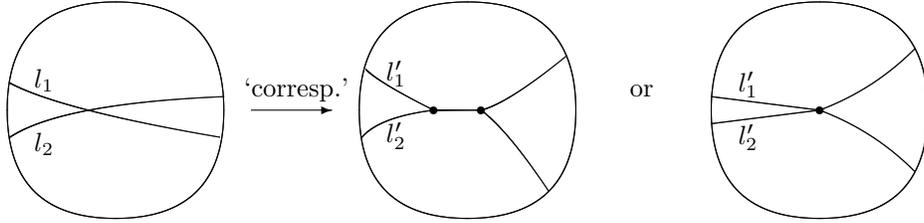
FIGURE 5. Transversal geodesics in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$ and two pictures which do not occur for corresponding $\tilde{g}_2$-geodesics.

Proposition 6.1 is implicit in the work of Otal, and has been reproven by Fathi and Hersonsky-Paulin (implicitly), and Croke-Fathi-Feldman (explicitly, see [CCF92, Lemma 1.5]). Each note that Otal's proof works in their situation with minor adjustments. The most significant adjustment comes in the case of cone singularities, and lies at the beginning of the argument. We describe this adjustment carefully here, then follow Otal and the subsequent presentations for the rest of the proof.

Let $\theta \in (0, \pi)$. Let $\theta \cdot v$ denote the rotation of v by θ (using the fact that S is oriented, this action commutes with that of Γ). Since $\hat{T}^1_{g_1}$ is full volume in $T^1_{g_1}$ and $\tilde{P}$ is countable, the set of (v, θ) such that both v and $\theta \cdot v$ lie in $\hat{T}^1_g$ is full measure in $T^1_{g_1} \times (0, \pi)$ using the volume on the tangent bundle and the Lebesgue measure on $(0, \pi)$.

Let $(v, \theta) \in \hat{T}^1_{g_1} \times (0, \pi)$ be a pair such that $\theta \cdot v \in \hat{T}^1_{g_1}$. Consider the $\tilde{g}_1$-geodesics l_v and $l_{\theta \cdot v}$ they generate in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$. Under the correspondence with $\tilde{g}_2$-geodesics, these correspond to geodesics l'_v and $l'_{\theta \cdot v}$ in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_2}$ (Corollary 5.9). Let $\theta'(v, \theta)$ be the $\tilde{g}_2$-angle between l'_v and $l'_{\theta \cdot v}$. Set $\theta'(v, 0) = 0$ and $\theta'(v, \pi) = \pi$. By Γ-equivariance of the above construction, $\theta'(v, \theta)$ induces a map, also denoted $\theta'(v, \theta)$, almost everywhere on $\hat{T}^1_{g_1}(S) \times [0, \pi]$. The measure zero set where $\theta'(v, \theta)$ is undefined does not make any difference to the rest of Otal's argument, which is measure-theoretic from this point.

Note that the work of Bankovic and Leininger is crucial (via Proposition 5.7) to defining $\theta'(v, \theta)$ almost everywhere in the presence of cone points. If we try to extend the definition to the problem cases shown in Figure 5 by, say, chosing one of the two natural 'candidates' for the angle between l'_v and $l'_{\theta \cdot v}$ then problems will arise when trying to prove Lemma 6.4 below. Hersonsky and Paulin also have to address the well-definedness problem in their use of Otal's work. They provide a separate proof of the fact that the correspondence takes non-singular geodesics to non-singular geodesics ([HP97, Lemma 4.15]) that uses their extensive work on the Möbius current (also in [HP97]). The Möbius current is defined using the crossratio and involves a fair bit of $\mathrm{CAT}(-1)$ technology, so it is not at all clear that something analogous can be done in the nonpositive curvature setting.

We now summarize the rest of Otal's argument to prove Proposition 6.1. For details of the proofs which are unaffected by our more general class of metrics, we refer to Otal's paper ([Ota90], particularly §2 and §3 through the proof of Lemme 8).

Let μ_g be the usual volume form on $T_g^1(S)$. Let $Vol(T_g^1(S))$ be the volume of $T_g^1(S)$ with respect to this measure.

Definition 6.3. *Let*

$$\Theta'(\theta) = \frac{1}{Vol(T_g^1(S))} \int_{T_g^1(S)} \theta'(v, \theta) d\mu_g.$$

Since $\theta'(v, \theta)$ is defined for almost all (v, θ), this integral is valid for almost all θ.

The following properties are fairly straightforward. The third assertion follows from the Gauss-Bonnet theorem with singularities.

Lemma 6.4. *Where defined, Θ' has the following properties:*
  (1) $\Theta' : [0, \pi] \to [0, \pi]$ *is increasing.*
  (2) *For all $\theta \in [0, \pi]$,*
$$\Theta'(\pi - \theta) = \pi - \Theta'(\theta).$$
  (3) *For all $\theta_1, \theta_2 \in [0, \pi]$ such that $\theta_1 + \theta_2 \in [0, \pi]$,*
$$\Theta'(\theta_1 + \theta_2) \geq \Theta'(\theta_1) + \Theta'(\theta_2).$$

Proof. See [Ota90, Prop. 6], or [HP97, Prop. 4.16]. For Otal, Θ' is a homeomorphism, but Hersonsky-Paulin note that only measurability is needed. $\qquad \square$

By the Jensen inequality, for any real-valued, strictly convex function F on $[0, \pi]$, for every θ,

$$(6.1) \qquad F(\Theta'(\theta)) \leq \frac{1}{Vol(T_g^1(S))} \int_{T_g^1(S)} F(\theta'(v, \theta)) d\mu_g$$

with equality for all F if and only if $v \mapsto \theta'(v, \theta)$ is constant. Integrating with respect to $\sin \theta d\theta$ and applying Fubini, we have

$$(6.2) \qquad \int_0^\pi F(\Theta'(\theta)) \sin \theta d\theta \leq \frac{1}{Vol(T_g^1(S))} \int_{T_m^1(S)} \left(\int_0^\pi F(\theta'(v, \theta) \sin \theta d\theta \right) d\mu_g.$$

Proposition 6.5. *For any convex function F,*

$$\int_0^\pi F(\Theta'(\theta)) \sin \theta d\theta \leq \int_0^\pi F(\theta) \sin \theta d\theta$$

Proof. Otal's proof of this result ([Ota90, Prop. 7]) can be extended to the 'measurable' case as noted in Hersonsky-Paulin Prop. 4.17. We note that in the course of this proof, Otal proves that

$$(6.3) \qquad \frac{1}{Vol(T_g^1(S))} \int_{T_m^1(S)} \left(\int_0^\pi F(\theta'(v, \theta) \sin \theta d\theta \right) d\mu_g = \int_0^\pi F(\theta) \sin \theta d\theta.$$

$\qquad \square$

Lemma 6.6. *Suppose that H is a measurable, increasing function on $[0, \pi]$, which is superradditive and commutes with the symmetry with respect to $\pi/2$. Suppose in addition that for any convex function F on $[0, \pi]$*

$$\int_0^\pi F(H(\theta)) \sin \theta d\theta \leq \int_0^\pi F(\theta) \sin \theta d\theta.$$

Then $H = Id$.

Proof. See [Ota90, Lemme 8] or [HP97, Lemma 4.18]. □

Proof of Proposition 6.1. With equation (6.2) and (6.3) we have that

$$\int_0^\pi F(\Theta'(\theta)) \sin\theta d\theta \le \int_0^\pi F(\theta) \sin\theta d\theta.$$

By Lemma 6.6, $\Theta' = Id$ so we have equality in the equation above. But this equation comes from integrating equation (6.1) over $[0,\pi]$ with respect to $\sin\theta d\theta$. So we must indeed have equality in equation (6.1) for almost all θ. As noted, this implies that $v \to \theta'(v,\theta)$ is constant for this full measure set of θ.

Returning to the definition of Θ' and using $\Theta' = Id$, we have for such θ that

$$\theta = \frac{1}{Vol(T_g^1(S))} \int_{T_g^1(S)} \theta'(v,\theta) d\mu_g.$$

As $\theta'(v,\theta)$ is constant in v, we must have $\theta'(v,\theta) = \theta$. This completes the proof. □

Corollary 6.7. $g_1 \in \mathscr{M}_{npc}(S)$.

Proof. This follows from the Gauss-Bonnet theorem and the angle correspondence given by Proposition 6.1. Indeed, any positive curvature for $\tilde{g}_1$ would be witnessed by some geodesic triangle with total angle sum $> \pi$, which cannot hold for the corresponding triangle under the nonpositively curved metric $\tilde{g}_2$.

To use Proposition 6.1 for this argument, we must ensure that the sides of this geodesic triangle are in $\hat{\mathcal{G}}_{\tilde{g}_1}^\circ$ and intersect at angles in the full-measure set provided by that proposition. Proposition 2.11 guarantees that almost all geodesics through any non-cone point are in $\mathcal{G}_{\tilde{g}_1}^\circ$, and $\mathcal{G}_{\tilde{g}_1}^\circ \setminus \hat{\mathcal{G}}_{\tilde{g}_1}^0$ has measure zero by our assumptions for Theorem 1. Using these facts it is easy to check that arbitrarily small geodesic triangles around any point of positive curvature answering the requirements of Proposition 6.1 can be constructed. □

7. Isometry on points of negative curvature

We now prove a version of Theorem 5.4 on the sets of strict negative curvature in (S, g_1) and (S, g_2). To do so, we follow ideas of Croke-Fathi-Feldman.

Definition 7.1. *Define a partial relation on $\tilde{S}$ by $p \sim p'$ if almost every non-singular $\tilde{g}_2$-geodesic through p' (with respect to the angular Lebesgue measure on the fiber of $T_{\tilde{g}_2}^1$ over p') is a bounded distance from non-singular $\tilde{g}_1$ geodesic through p.*

Following Croke-Fathi-Feldman we have the following results.

Lemma 7.2 ([CCF92], Lemma 2.3). *The relation $\sim$ is the graph of a bijective function F_{neg} between its domain $\tilde{D}_1$ and range $\tilde{D}_2$. $\tilde{D}_1$ and $\tilde{D}_2$ are Γ-invariant and F_{neg} is Γ-equivariant.*

Proof. We need to prove the following. If $p \sim p_1'$ and $p \sim p_2'$, then $p_1' = p_2'$. That F_{neg} is a bijection follows from reversing the roles of $\tilde{g}_1$ and $\tilde{g}_2$. (Recall that by Corollary 6.7 both metrics are now in $\mathscr{M}_{npc}(S)$.)

We are given that almost every non-singular $\tilde{g}_2$-geodesic through p_i' is a bounded distance from a non-singular $\tilde{g}_1$-geodesic through p. We note that two distinct $\tilde{g}_2$-geodesics through p_i' cannot correspond to the same $\tilde{g}_1$-geodesic through p as they have different endpoints in $\partial_\infty(\tilde{S})$. We then see that the map from $T_{\tilde{g}_2}^1(p_i')$ to $T_{\tilde{g}_2}^1(p)$

induced by the correspondence of these non-singular geodesics is a strictly increasing map with respect to the angular order on these spaces. A strictly increasing map defined almost everywhere is continuous almost everywhere, and by applying the same argument with the roles of p and p'_i reversed we see that the map is in fact extendable to a continuous function from the circle of directions at p'_i to the circle of directions at p.

Using this, we see that almost every non-singular $\tilde{g}_2$-geodesic through p'_1 is a bounded distance from a non-singular $\tilde{g}_2$-geodesic through p'_2. These pairs of geodesics must bound flat strips, but by Proposition 2.13 we know this is impossible.

$\square$

Let $\tilde{U}_i = \{p \in (\tilde{S} \setminus \tilde{P}, \tilde{g}_i) : \kappa_{\tilde{g}_i}(p) < 0\}$ be the set of non-cone points where the $\tilde{g}_i$-curvature is strictly negative. This is an open subset of $\tilde{S}$.

Lemma 7.3 ([CCF92], Lemma 2.4). *$\tilde{U}_1$ is in the domain $\tilde{D}_1$ of F_{neg}.*

Proof. Let $p \in \tilde{U}_1$. Note that no $\tilde{g}_1$-geodesic through p bounds a flat strip. Pick any two non-singular geodesics l_1 and l_2 through p. Then l_1 and l_2 are in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_1}$. Let l'_i be corresponding $\tilde{g}_2$-geodesics, and let their intersection be p' – it is unique by Proposition 6.1. We also note that l'_i are in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_2}$ by Corollary 5.9.

Now let l'_3 be a geodesic in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_2}$ through p'. Note that almost every geodesic through p' is in $\hat{\mathcal{G}}^{\circ}_{\tilde{g}_2}$, since, by the argument of Proposition 2.13 only countably many flat strips can pass through any point.

Now consider the geodesic triangle formed by l_1, l_2, and l_3. By the angle correspondence of Proposition 6.1, it has angle sum π, and since $\tilde{g}_1$ has negative curvature at p, the triangle must degenerate to the point p by Gauss-Bonnet with singularities. Thus l_3 passes through p and we have proven that $p \sim p'$, as desired. $\square$

Remark 7.4. Note that F_{neg} works in the same way F_c of section 5 does – by taking p to the common intersection of $\tilde{g}_2$-geodesics corresponding to a full measure set of $\tilde{g}_1$-geodesics passing through p.

Proposition 7.5. *Let $g_1, g_2 \in \mathscr{M}_{npc}(S)$ with the same marked length spectrum. Then the map F_{neg} described above is a Γ-equivariant bijective isometry between $\tilde{U}_1$ and $\tilde{U}_2$.*

Proof. Let $p_1, p_2 \in \tilde{U}_1$, and let k be the $\tilde{g}_1$-geodesic segment between them. Consider the set $\mathcal{G}_{\tilde{g}_1}(k)$ of all $\tilde{g}_1$-geodesics intersecting k transversally. $F_{neg}(p_1)$ and $F_{neg}(p_2)$ lie in at most countably many flat strips. So it is not hard to see that all but countably many $\{a, b\} \in \partial_{\tilde{g}_1}(\mathcal{G}_{\tilde{g}_1}(k))$ correspond to elements of $\partial_{\tilde{g}_2}(\mathcal{G}_{\tilde{g}_2}(k'))$ where k' is the geodesic segment connecting $F_{neg}(p_1)$ and $F_{neg}(p_2)$, and vice versa. Then, just as in the last step of the proof of Theorem 5.4,

$$
\begin{aligned}
d_{\tilde{g}_1}(p_1, p_2) &= L_{g_1}(\partial_{\tilde{g}_1}(\mathcal{G}_{\tilde{g}_1}(k))) \\
&= L_{g_2}(\partial_{\tilde{g}_2}(\mathcal{G}_{\tilde{g}_2}(k'))) \\
&= d_{\tilde{g}_2}(F_{neg}(p_1), F_{neg}(p_2)).
\end{aligned}
$$

We now know that $F_{neg} : \tilde{U}_1 \to \tilde{S}$ is a metric isometry; it is clearly Γ-invariant. By considering this metric isometry applied to short geodesic segments in $\tilde{U}_1$, it is

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easy to see that it must in fact be a Riemannian isometry, and a homeomorphism onto its image. Therefore it must take $\tilde{U}_1$ into $\tilde{U}_2$. Reversing the roles of the two metrics proves it is bijective, completing the proof. $\square$

8. Building the full isometry

In sections 5 and 7 we built Γ-equivariant isometries $F_c : cone(\tilde{g}_1) \to cone(\tilde{g}_2)$ and $F_{neg} : \tilde{U}_1 \to \tilde{U}_2$ between the cone points and points of negative curvature, respectively. We first note that both of these maps are defined in the same way – by showing that the correspondence between $\tilde{g}_1$-geodesics and $\tilde{g}_2$-geodesics takes (a full measure set of) geodesics through a point p to (a full measure set of) geodesics through $F_-(p)$. The proof that the maps are isometries is the same in each case. Therefore, these maps can be combined into a single isometry:

Proposition 8.1. $F' := F_c \cup F_{neg} : cone(\tilde{g}_1) \cup \tilde{U}_1 \to cone(\tilde{g}_2) \cup \tilde{U}_2$ is a Γ-equivariant isometry.

What remains is to extend this isometry to the set of points of curvature zero. To do so we follow ideas of both Croke-Fathi-Feldman and Bankovic-Leininger.

The following two lemmas will be useful.

Lemma 8.2. Let $p_1, p_2, q_1, q_2 \in cone(\tilde{g}_1) \cup \tilde{U}_1$. If the $\tilde{g}_1$-geodesic segments $[p_1, p_2]$ and $[q_1, q_2]$ intersect in their interiors, then the $\tilde{g}_2$-geodesic segments $[F'(p_1), F'(p_2)]$ and $[F'(q_1), F'(q_2)]$ intersect in their interiors.

Proof. Recall that $F'(p)$ is the common intersection point of $\tilde{g}_2$-geodesics corresponding to almost every $\tilde{g}_1$-geodesic through p. From this it is immediate that any geodesic l through p_1 and p_2 corresponds to a geodesic through $F'(p_1)$ and $F'(p_2)$. Then the configuration of geodesics in $\hat{\mathcal{G}}_{\tilde{g}_1}^-$ in the left half of Figure 6 must correspond to a configuration as in the right half, where h' and l' intersect between l'_1 and l'_2.
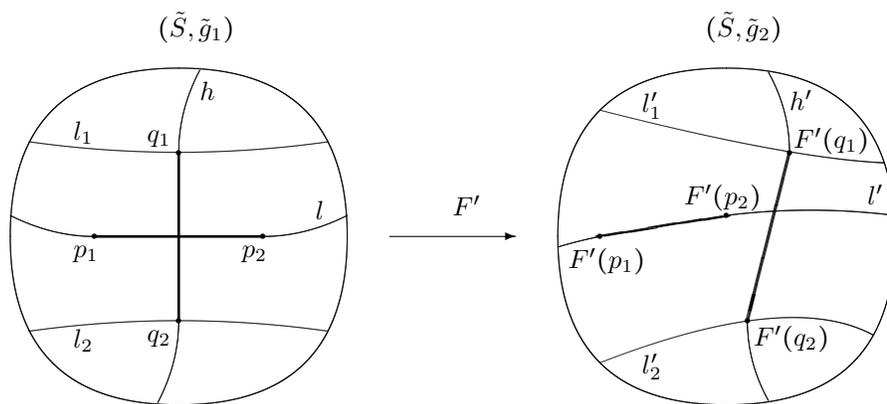

FIGURE 6. F' applied to intersecting segments.

(Note that such a configuration can be drawn, with l, l_1 and l_2 having distinct endpoints, because none of the p_i, q_i can lie in the interior of a flat strip.)

Reversing the roles of p_i and q_i, we see that Figure 6 is inaccurate as drawn. In fact h' must cross l' between $F'(p_1)$ and $F'(p_2)$, proving the lemma.

$\square$

Lemma 8.3. *Let p_1, p_2, p_3 and q be points in $cone(\tilde{g}_1) \cup \tilde{U}_1$ such that q lies in the interior of the $\tilde{g}_1$-geodesic triangle formed by p_1, p_2, and p_3. Then $F'(q)$ lies in the interior of the $\tilde{g}_2$-geodesic triangle formed by $F'(p_1), F'(p_2)$, and $F'(p_3)$.*
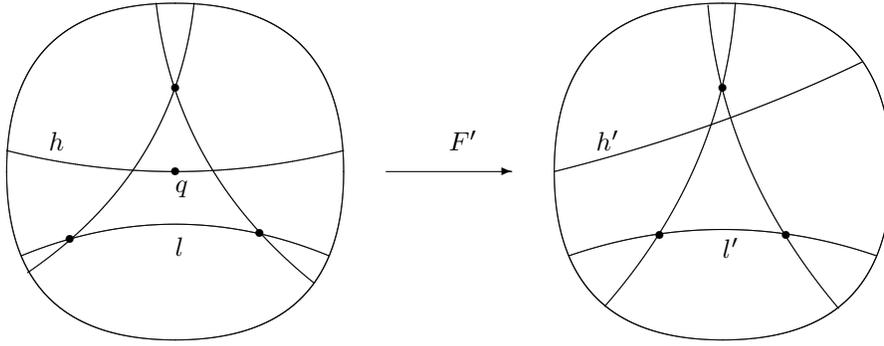

FIGURE 7. F' applied to geodesic triangles.

Proof. Figure 7 shows that $F'(q)$ must lie in the open half-space bounded by l' containing $\Delta(F'(p_1), F'(p_2), F'(p_3))$. Running this argument for all three sides of $\Delta(p_1, p_2, p_3)$ proves the lemma. (Again, the configuration in this figure can be drawn since q does not belong to the interior of any flat strip and so h can be taken to have distinct endpoints from l.) $\square$

Note that we can assume that $cone(g_1)$ (and hence $cone(g_2)$ by Theorem 5.4) are non-empty – otherwise Croke-Fathi-Feldman applies. Then we may take a Γ-invariant geodesic triangulation $\tilde{\tau}$ of $(\tilde{S}, \tilde{g}_1)$ such that the vertices are precisely the cone points. (This is an easy construction). The lemmas above show that F' sends this triangulation to a triangulation of $(\tilde{S}, \tilde{g}_2)$:

Lemma 8.4. *Let $F'(\tilde{\tau})$ be the collection of $\tilde{g}_2$-geodesic segments $\{[F'(p_1), F'(p_2)] : [p_1, p_2] \in \tau\}$. Then $F'(\tilde{\tau})$ is a Γ-invariant triangulation of $(\tilde{S}, \tilde{g}_2)$ with vertex set precisely $cone(\tilde{g}_2)$. Further, if $q \in cone(\tilde{g}_1) \cup \tilde{U}_1$ belongs to a triangle T of $\tilde{\tau}$, then $F'(q) \in F'(T)$, where $F'(T)$ is the triangle formed by the F' images of the vertices of T.*

Proof. That $F'(\tilde{\tau})$ is Γ-invariant with vertex set $cone(\tilde{g}_2)$ is clear. That it preserves containment in triangles is an immediate application of Lemma 8.3.

Every segment in $\{[F'(p_1), F'(p_2)] : [p_1, p_2] \in \tilde{\tau}\}$ belongs to two 3-cycles in $F'(\tilde{\tau})$, thought of as an abstract graph, on opposite sides of that edge, since this is true for τ. By Lemma 8.2 these segments do not intersect away from the vertices.

Then the geometric realization of this graph in the simply connected $\tilde{S}$ is an infinite, locally finite, planar graph, all of whose edges belong to two triangles on opposite sides of the edge. Therefore it is a triangulation. $\qquad\square$

To complete the proof of Theorem 1 we need to extend F' to a Γ-equivariant isometry on all of $(\mathring{S}, \tilde{g}_1)$.

Proof of Theorem 1. Fix a Γ-invariant geodesic triangulation $\tilde{\tau}$ of $(\tilde{S}, \tilde{g}_1)$ as above.

Let $\mathcal{T}_0(\tilde{\tau})$ be the set of all triangles of $\tilde{\tau}$ which have $\tilde{g}_1$-flat interior. Since F' respects containment of points with negative curvature in triangles, the induced map from triangles of $\tilde{\tau}$ to triangles of $F'(\tilde{\tau})$ takes $\mathcal{T}_0(\tilde{\tau})$ bijectively to $\mathcal{T}_0(F'(\tilde{\tau}))$. Let $V_1 = \{p \in \bar{T} : T \in \mathcal{T}_0(\tilde{\tau})\}$ and $V_2 = \{p \in \bar{T} : T \in \mathcal{T}_0(F'(\tilde{\tau}))\}$. Since F' is an isometry on the vertices of these Euclidean triangles, it can be extended to $F'_0 : cone(\tilde{g}_1) \cup U_1 \cup V_1 \to cone(\tilde{g}_2) \cup U_2 \cup V_2$ as a Riemannian isometry away from the cone points. It is clear that F'_0 is well-defined on any points at the boundary of two such triangles, and that F'_0 is Γ-equivariant.

If $\tilde{g}_1$ is Euclidean away from cone points, this completes the proof. This is similar to, but not exactly, the argument of Bankovic-Leininger.

Now let $\mathcal{T}_<(\tilde{\tau})$ be the set of all triangles of $\tilde{\tau}$ for which there is some point in $Int(T)$ at which the $\tilde{g}_1$-curvature is strictly negative. To extend F'_0 to these triangles, we use the approach of Croke-Fathi-Feldman. $\mathcal{T}_<(\tilde{\tau})$ is certainly Γ-invariant and F' takes this collection of triangles to $\mathcal{T}_<(F'(\tilde{\tau}))$. Pick, for each triangle T in $\mathcal{T}_<(\tilde{\tau})$ a point $p_T^* \in Int(T)$ at which $\kappa(p_T^*) < 0$ in a Γ-invariant way. Let $V_T \subset T_{p_T^*}\tilde{S}$ be such that $\exp_{p_T^*}^{\tilde{g}_1}(V_T) = \bar{T}$. This set exists since there are no cone points in the interior of T and the exponential map is injective as we are in nonpositive curvature. Then, following Croke-Fathi-Feldman, we define

$$
\begin{aligned}
F_T : \quad (T, \tilde{g}_1) \quad &\to \quad (\tilde{S}, \tilde{g}_2) \\
p \quad &\mapsto \quad (\exp_{F'_0(p_T^*)}^{\tilde{g}_2}) \circ (DF'_0)_{p_T^*} \circ (\exp_{p_T^*}^{\tilde{g}_1})^{-1}(p).
\end{aligned}
$$

First, we claim that F_T extends F'_0, that is, if $q \in \bar{T} \cap (cone(\tilde{g}_1) \cup \tilde{U}_1)$, then $F_T(q) = F'_0(q)$. Let $c(t)$ be the unit-speed $\tilde{g}_1$-geodesic connecting p_T^* to q with $c(0) = p_T^*$. Then $d_{\tilde{g}_1}(c(t), q) = d(p_T^*, q) - t$. For sufficiently small $|t|$, $c(t)$ lies in $\tilde{U}_1$ since $c(0)$ does. Thus, for sufficiently small $|t|$, we can use the fact that F'_0 is an isometry and we have

$$
d_{\tilde{g}_2}(F'_0(c(t)), F'_0(q)) = d_{\tilde{g}_2}(F'(p_T^*), F'(q)) - t.
$$

This implies that $F'_0(c(t))$ lies along the geodesic segment connecting $F'_0(p_T^*)$ and $F'_0(q)$ for at least small values of $|t|$. In particular, $(DF'_0)_{p_T^*}(\dot{c}(0))$ is tangent to the geodesic from $F'_0(p_T^*)$ to $F'_0(q)$. Then, using the definition of F_T and the fact that F'_0 is an isometry, it is easy to see that $F_T(q) = F'_0(q)$.

Second, F_T must preserve curvature, since we now know it agrees with the isometry F'_0 on points of negative curvature and cone points, and all other points have zero curvature.

Third, we claim that $F_T|_{Int(T)}$ is a Riemannian isometry onto its image. At points in $\tilde{U}_1$ this follows from the fact that it is a (metric) isometry. Let q be a point in $\tilde{U}_1^c$ and $v \in T_q\tilde{S}$. If v is tangent to the geodesic connecting p_T^* and q, then it is easy to see from the definition of F_T that $|(DF_T)_q(v)|_{\tilde{g}_2} = |v|_{\tilde{g}_1}$. Now assume that v is normal to the geodesic from p_T^* to q. There is a unique Jacobi field along

this geodesic with value 0 at p_T^* and value v at q. Since any Jacobi field arises from a variation of geodesics, DF_T takes this Jacobi field to a Jacobi field along the geodesic from $F_T(p_T^*)$ to $F_T(q)$. By the second point above, the curvatures along the geodesic segments $[p_T^*, q]$ and $[F'(p_T^*), F'(q)]$ agree, so by the Jacobi equation, $|(DF_T)_q(v)|_{\tilde{g}_2} = |v|_{\tilde{g}_1}$. This proves the claim.

Fourth, we claim that F_T has its image in the geodesic triangle $F'(T)$ from the triangulation $F'(\tilde{\tau})$. Since F_T is a Riemannian isometry on its interior, and since the geodesics bounding T can be approached by sequences of geodesics in $Int(T)$, we see that F_T takes T to a geodesic triangle. From the first claim, we know F_T takes the vertices of T to the vertices of $F'(T)$. But there is only one geodesic triangle with these vertices – namely $F'(T)$. Reversing the roles of $\tilde{g}_1$ and $\tilde{g}_2$ shows that F_T is bijective onto its image.

We now let $F = \bigcup_{T \epsilon \tilde{\tau}} F_T$. It is clear from its construction that F is Γ-equivariant and extends F_0'. The fact that each F_T maps the geodesic edges to the geodesic edges of $F'(T)$ preserving arc-length implies that F is well-defined along the edges of $\tilde{\tau}$. The fact that F_T is a Riemannian isometry on the open dense subset consisting of the interiors of all the triangles in $\tilde{\tau}$ and an isometry on the cone point sets, together with the fact that distance for $\tilde{g}_1$ or $\tilde{g}_2$ is realized by lengths of shortest paths then easily implies that F is an isometry.

To prove that the map F induces on $(S, g_1) \to (S, g_2)$ is isotopic to the identity, we note that by Γ-equivariance, the extension of F to $\partial^\infty(\tilde{S})$ is the identity, so the map F induces on Γ is the identity, proving the result.

This completes the proof of Theorem 1.

$\square$

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